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SOLVING LINEAR FIRST ORDER DIFFERENTIAL EQUATION

The simplest linear first order equation is :

$$y' = ky$$

where k is constant. It is linear because its terms dependent on y and y’ in a linear way, and it is first order because the first derivative of y is the highest order derivative involved.

Observation:

- The equation has the trivial solution

$$y(t) = 0 \Rightarrow \text{for all } t$$

- If y is a solution and c is a constant, then c.y is also a solution.

  in particular, -y is also a solution.

$$(cy)' = k(cy) \Rightarrow cy' = kcy \underset{c \neq 0}{\Rightarrow} y' = ky$$

So let’s find the positive solutions. The equation is separable because we can rewrite (using t as the independent variable)

$$\frac{dy}{dt} = ky$$

In the form

$$\frac{dy}{y} = kdt$$

Which allows anti-differentiation of each side, resulting in:

$$\ln y = kt + c$$

where c is an arbitrary constant, or parameter.

$$y = e^{kt+c} = e^{c}e^{kt} => y = ce^{kt} = y(0)e^{kt}$$

So the result is that :

$$y' = ky => y(t) = y(0)e^{kt} = y(t0)e^{k(t-t0)}$$

$$y' + ky = 0 => y(t) = y(0)e^{-kt} = y(t0)e^{-k(t-t0)}$$

Now let’s look at a somewhat more general case:

$$y' + py = 0$$

Where p is a given function of the independent variable.

The equation is still separable:

$$\frac{dy}{y} = -p(t)dt \Rightarrow \ln y = -\int p(t)dt + c$$

$$y = ce^{-\int p(t)dt}$$

Example: find a general solution of y’+2ty=0

$$\frac{dy}{y} = -2t\,dt$$

$$\ln y = -t^2 + c$$

$$y = e^c e^{-t^2}$$

$$\therefore y = ce^{-t^2}$$

Example: find a general solution of y'+ ysinx = 0

$$\frac{dy}{y} = -\sin x dx$$

$$\ln y = \cos x + c$$

$$y = ce^{\cos x}$$

Example: find a general solution of $y' + \frac{3}{5-2t} y = 0$

$$\frac{dy}{y} = -\frac{3dt}{5-2t}$$

$$\ln y = -3\int \frac{1}{u}\frac{-du}{2} = \frac{3}{2}\ln u + c$$

$$\ln y = \ln(5-2t)^{3/2} + c$$

$$y = c(5-2t)^{3/2}$$

$$assume:$$

$$u = 5-2t$$

$$du = -2dt$$

The non-homogeneous case

$$y' + py = q$$

Observation : the difference between two solution will satisfy the corresponding homogeneous or reduced equation.

$$y' + py = 0$$

Whose general solution we've just learned is

$$y = ce^{-\int p}$$

So suppose we can find some particular solution: call it z. then y is any other solution, we know that:

$$y - z = ce^{-\int p} \quad \text{for some c}$$

Therefore, every solution y has the form

$$y = ce^{-\int p} + z$$

This “superposition” method is very convenient way to solve the equation provided we can find a particular solution z without too much effort, such as when p and q are constant functions.

Example: find a general solution of $y' + 3y = 2$

1. $z=\frac{2}{3}$ Is a particular solution

2. The general solution of the reduced equation

$$y'+3y=0 \quad \text{is} \quad y(x)=ce^{-3x}$$

3. Therefore the general solution of the original equation is:

$$y(x)=ce^{-3x}+\frac{2}{3}$$

Using an integrating factor, returning to the general case

$$y'+py=q$$

One might think of the product rule and surmise that if the left hand side were multiplied by an appropriate function. It might be recognizable as the derivative of a certain product.

That is in fact true, and an appropriate function is $e^{\int p}$ where the symbol $\int$p represent any anti-derivative of p. we call this function as an integrating factor.

For the differential equation, by multiplying both side with the integrating factor gives:

$$e^{\int p} y' + p e^{\int p} y = q e^{\int p}$$

The left hand side is now the derivative of the product $e^{\int p} y$:

$$(e^{\int p} y)' = q e^{\int p}$$

Next we simply integrate both side and then divide by $e^{\int p}$

Here’s the procedure in steps:

1. anti-differentiate p, obtaining ∫p (take c=0)
2. Form and simplify $e^{\int p}$, the integrating factor.
3. Multiply the equation by $e^{\int p}$, obtaining

$$e^{\int p}y' + pe^{\int p}y = qe^{\int p}$$

and check that it’s the same as

$$(e^{\int p}y)' = qe^{\int p}$$

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4. Integrate and divide by $e^{\int p}$. Showing the constant of integration c for emphasis, the proceeds generically as follows;

$$e^{\int p} y = \int (q e^{\int p}) + c$$

$$y = e^{-\int p} \int (q e^{\int p}) + c e^{-\int p}$$

Example : find a general sol of $y'+3y=2e^{-x}$

1. p=3, sp ∫p = 3x
2. An integrating factor is e^{3x}
3. Next multiply by the integrating factor

$$e^{3x}y'+3e^{3x}y=2e^{-x}e^{3x}$$

$$(e^{3x}y)'=2e^{2x}$$

4. Then integrate and solve for y:

$$e^{3x}y=e^{2x}+c$$

$$y=e^{-x}+ce^{-3x}$$

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Example: find a general solution of $y' + \dfrac{3y}{5-2t} = 1$

1. What is the p?

$$\int \frac{3dt}{5-2t} = 3\int \frac{1}{u}\left(-\frac{du}{2}\right) = -\frac{3}{2}\ln(5-2t) = \ln(5-2t)^{-3/2}$$

2. What is the an integrating factor?

$$e^{\ln(5-2t)^{-3/2}} = (5-2t)^{-3/2}$$

3. Multiply the integrating factor.

$$(5-2t)^{-3/2}y' + 3(5-2t)^{-5/2}y = (5-2t)^{-3/2}$$

$$[(5-2t)^{-3/2}y]' = (5-2t)^{-3/2}$$

4. Integrate and solve for y.

$$(5-2t)^{-3/2}y = -\frac{2}{2}(5-2t)^{-1/2} + c$$

$$y = 5 - 2t + c(5-2t)^{3/2}$$

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Example: find a general solution of $y'+y\tan x=\sin x$

1. What is P?

$$\int \tan x dx = \ln\left|\sec x\right|$$

2. What is the an integrating factor?

$$e^{\ln\sec x} = \sec x$$

3. Multiply the integrating factor.

$$y'\sec x + y\sec x\tan x = \tan x$$
$$(y\sec x)' = \tan x$$

4. Integrate and solve for y.

$$y\sec x = \ln\left|\sec x\right| + c$$
$$y = (\ln\left|\sec x\right| + c)\cos x$$

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Summary

The homogeneous equation y'+ py = 0 is separable. Its general solution is :

$$y = ce^{-\int p}$$

If p is constant, then this becomes $y(t) = ce^{-pt}$

The non-homogeneous equation $y' + py = q$

Is not separable unless p and q are constants. When p and q are constants, the general solution is

$$y(t) = ce^{-pt} + \frac{q}{p}$$

In all cases, multiplying by the integrating factor $e^{\int p}$ yield the equation

$$(e^{\int p} y)' = e^{\int p} q$$

Which can be solved by anti-differentiating both sides.

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To be Continue…

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